Fourier Series

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Purpose:

Verify the Fourier Series of the square wave.

Equipment Required:

- 1 - Agilent 54622A Deep Memory Oscilloscope or Agilent 54600B Oscilloscope
- 1 - Agilent 34401A Digital Multimeter
- 1 - Agilent 33120A Waveform Generator
- 1 - Protoboard
  Passive elements as required

Prelab:

1. The spectral plot

   The Fourier Series represents a periodic waveform as an infinite series of harmonically related sinusoids. Because the Fourier series contains only discrete frequencies, the spectral content of a waveform can be represented as a "stick" plot like Fig. 1. In a stick plot each sinusoidal component of the waveform is represented by a vertical line (or stick) on a plot of the normalized signal magnitude versus frequency. The height of the line represents the magnitude of the contribution from that particular frequency. The location of the line along the horizontal axis identifies its frequency. The stick plot of Fig. 1 represents the Fourier series for a square wave (Eq. 1) with f_o = 1.1 kHz.
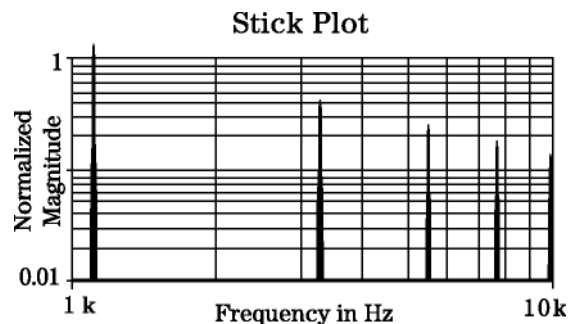

Figure 1

$$f(t) = \frac{4A}{\pi} \left[\sin 2\pi f_0 t + \frac{1}{3} \sin 2\pi 3 f_0 t + \frac{1}{5} \sin 2\pi 5 f_0 t + ... \right] \qquad (1)$$

The *spectrum analyzer* is a bench-top instrument that will display a plot of signal magnitude versus frequency. While its display is not quite a "stick" plot, the spectrum analyzer will identify the frequency content, or spectral content, of the input signal. The spectrum analyzer can be modeled by a very selective, high-Q bandpass filter, whose center frequency f_c is swept across a range of frequencies specified by the user. The output of this variable-center-frequency filter is plotted on the Y axis, and frequency is plotted on the X axis.

A plot of the normalized output versus input frequency of a high-Q filter with a square wave input looks very much like the display of the spectrum analyzer, except the peaks are in reverse order. The graph of the normalized output is started an order of magnitude or more below the center frequency of the filter f_c. As the frequency of the square-wave input to the high-Q filter is increased, the Fourier harmonics move into and then out of the passband of the high-Q filter. When one of the harmonics is centered inside the passband of the filter, the amplitude of the output signal reaches a local maximum.

In this laboratory exercise you will use a plot of the normalized output to verify that the spectral content of a square wave agrees with the spectral content predicted by the Fourier series of the square wave. The notation f_c will be used in this lab exercise to indicate the center frequency of the filter. This notation will avoid possible confusion with f_o, which denotes the fundamental frequency of the square-wave signal.

2. Design a high-Q filter

Design a high-Q bandpass filter with a center frequency f_c of 10 kHz and a Q≥10. One solution is to use an RLC circuit. Section 13-6 in the text* provides details on the circuit quality factor Q for RLC resonant circuits. Verify that your design meets the center frequency f_c, and quality factor Q specifications by analyzing your circuit with a CCA program.
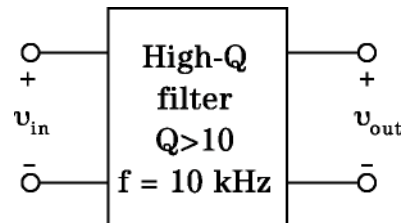

Figure 2

Procedure

1. Build the high-Q filter

Build and test your design on a protoboard. With the function generator set to produce a sine wave, measure the center frequency f_C of your filter. Collect data from 1 kHz to 100 kHz, then plot the gain portion of a Bode diagram. From this plot find the Q of your filter.

2. Collect the data

The goal is to verify that the spectral content of the square wave agrees with that predicted by the Fourier series. The peaks shown in Fig. 3, labeled f_o, f_1, etc. are due to harmonics of square wave signals inserted into a high-Q filter.

a. In your journal, prepare a one page data table with five columns and enough rows to record about 30 data points. Label the columns: source frequency, description, v_{in}, v_{out} and gain.
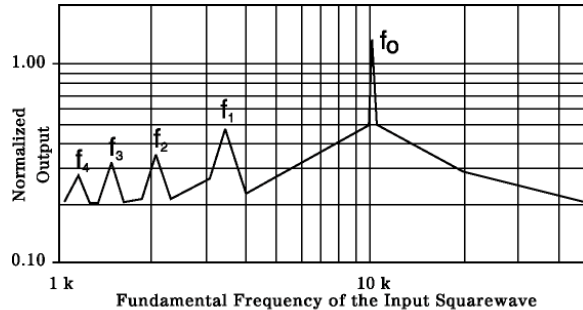

Figure 3

b. Set the function generator to produce a 1-kHz square wave with an amplitude of 5 V and no DC offset. Configure the scope to display v_{in} and v_{out}. Set the time base on the scope so that two complete cycles are visible on the CRT. The unusual shape of the output waveform v_{out} is the result of filtering by the high-Q filter circuit. The shape of the output waveform is not a concern in this exercise, since only the input frequency and the input and output peak-to-peak amplitudes will be measured. While observing v_{out}, slowly increase the frequency of v_{in} up through 10 kHz. Notice that the peak-to-peak amplitude of v_{out} goes through several local minimums and maximums. Finally, v_{out} reaches its largest value at an input frequency equal to the center frequency of the high-Q filter.

c. Begin acquiring data by locating the frequencies where local maximums occur in v_{out}. Starting at 1 kHz, slowly increase the frequency until the amplitude of v_{out} peaks and then begins to roll off. This should occur at $f_C/9 \approx 1.11$ kHz. Tune the frequency back and forth until you are satisfied v_{out} is at a local maximum. Leave two blank lines at the top of the data table. Carefully measure and on the third line of the data table record the frequency of this first maximum of v_{out}. At this same frequency measure and record the peak-to-peak values of v_{in} and v_{out}. In the description column enter the notation "max @ $f_C/9$." Now increase the frequency until you have located the next maximum in v_{out}. Leave five blank lines in the data table between the first data point and this new one, and measure and record V_{PP} for v_{in}, v_{out} and the frequency for the second maximum. The measured frequency for this data point should be equal to the center frequency of the high-Q filter divided by 7. Enter "max @ $f_C/7$" in the description column. Follow this procedure — leave five blank lines then measure and record data at each maximum of v_{out} — until you have located and recorded each of the maximums up to and including an input frequency equal to the center frequency of the high-Q filter, f_C.

d. Now go back and record five data points between each of the maximums. At a frequency approximately midway between each of the measured maximums, measure and record v_{in}, v_{out} and the frequency of v_{in}. This point equates to a minimum in v_{out}, but it may be difficult to tell this from the oscilloscope display. Measure and record v_{in}, v_{out} and the input frequency at two more frequencies below this midpoint. These measurements should be spaced approximately equi-distant between the adjacent maximum and the midpoint. Repeat this procedure at two equally space frequencies between the midpoint and the next higher-frequency maximum. Collect data in this manner until all the rows between the existing maximums are filled in.

e. Collect two data points below the $f_C/9$ maximum at frequencies of 1 kHz and 500 Hz.

f. Verify that there are no additional maximums in v_{out} above 10 kHz. Do this by starting at an input frequency of f_C (approx 10 kHz) and while observing v_{out}, slowly increase the frequency. Record in your journal whether any maximums occur above 10 kHz.

g. Collect two more points above f_C, one at 12 kHz and another at 20 kHz.

Conclusion

1. Plot the gain vs. frequency

   Key your data into a spreadsheet program, and use the spreadsheet to compute the normalized output for each data point. Generate a graph of the normalized output vs. frequency using a log-log grid. The graph of the magnitude of the normalized output should look something like Fig. 3, except with better resolution. Print out the graph and tape it into your lab journal. For each local maximum on this plot, compute the ratio of the center frequency of the filter to the input frequency, f_C/f.

2. Verify the spectral content

   Compute the error between the ratio f_c/f computed in part 1 of the Conclusion and the expected harmonic number for that peak in v_{out}. Comment on whether the frequency measured at each of the maximums in v_{out} support the claim that they are the harmonics found in the Fourier series of the squarewave.